Computerized Symbolic Computation on a Sixth-order Model for Liquid Waves in the Presence of Surface Tension or a Floating Ice

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Computerized symbolic computation reflects the rapid expansion of computer sciences in various fields of science and engineering, while the studies on the liquid surfaces for rivers, oceans, aviation kerosene, liquid propellant for rockets, etc., are of current interest. In the presence of surface tension or sea ice, and with symbolic computation, the Hărăgus-Courcelle-Il'ichev model for surface liquid waves is hereby investigated. Several similarity reductions are presented, some of which are explicitly written out as exact analytic solutions having their rational expressions with respect to the dimensionless spatial variables of the model.

Key words: Symbolic Computations; Liquid Waves in the Presence of Surface Tension or a Floating Ice; Similarity Reductions; Exact Analytic Solutions

Investigations on the liquid surfaces are of interest, such as those for rivers, oceans, aviation kerosene and liquid propellant for rockets. In the presence of surface tension or sea ice, the flexural-gravity or water-ice waves are of practical value in such studies as those on the damage to offshore constructions by floating ice sheets, ice growth on structures and stress control for the facilities built upon the ice. Readers interested in those topics are referred to [1 – 7]. On the other hand, for the gravity-capillary case, the experimental data and theoretical results are relatively hard to be compared [8, 9].

Among the intesting ones, the Hărăgus-Courcelle-Il'ichev model for the gravity-capillary and flexural-gravity waves,

$$u_{xt} + (u\,u_x)_x + s\,u_{xxxx} + u_{xxxxxx} + u_{yy} = 0, \quad s = \pm 1 \quad (1)$$

has recently been proposed [4, 5, 10], which is (2+1)-dimensional and 6th-ordered, where x, y and t are dimensionless spatial and temporal variables and u is a dimensionless surface deviation. Equation (1) generalizes the Kadomtsev-Petviashvili (KP) equation to the presence of higher order dispersive effects (caused either by sea ice or to surface tension), and the fifth order Korteweg-de Vries (KdV) equation to (2+1) dimensions. For the standard KP equation, plane soliton solutions may be unstableor stable depending on the sign of the dispersion (hereby, sign of s). Also, [11] reports that the higher order dispersive terms are related to the radiation of the KdV solitons. Advanced topics on the KP equation can be found, e. g., in [12 – 17], while on the 5th-order KdV equation, e. g., in [4, 18 – 20].

With respect to (1) for dimensionless bond numbers $b > \frac{1}{3}$, and for ice plates with large initial tensions, $s = -1$; otherwise $s = 1$. Derivations of (1) have been given in [4] for those different cases, characterized by different values of s. See also [5, 10] for details. [4] presents several approximate analytic solutions, subject to either periodic or Dirichlet boundary conditions in the direction transverse to the propagation (i. e., the y direction), which are the travelling solitary waves having damped oscillations and propagating in a channel (along the x-axis). The instability treatment [5] indicates that travelling periodic waves subject to x-homogeneous y-axis perturbations, also analytic but approximate, are found to decay into a sequence of parallel wave guides or channels [5], each of which represents a wave propagating along the x axis but localized in the y direction. The details on the types of wave propagation, including the stability of solutions and the presence of solitary wave solutions, can be seen in [4, 5, 10]. For $s = -1$, [9] proposes some possibly observable effects for soliton-like liquid wave propagation in the presence of surface tension and [6] reports the instability and collapse of waveguides on the water surface under the ice cover. For both $s = \pm 1$ cases, [7] presents an auto-Bäcklund transformation and types of the solitonic and other exact analytic solutions, some possibly observable effects for future experiments and a possible way to explain the regular structure of the open-sea ice break observations. To our knowledge, no similarity solutions to (1) have appeared as yet.

Symbolic computation increases drastically the ability of a computer to deal exactly and algorithmically with the analytic expressions [12, 17, 23].

In this paper we will make use of symbolic computation in the sense of [22] to construct certain similarity reductions and rational solutions of (1).

Firstly, similar to [22], we can show that it is sufficient to seek the similarity reductions of (1) in the form of

$$u(x,y,t) = \alpha(x,y,t) + \beta(x,y,t)\, w[z(x,y,t)], \qquad (2)$$

rather than the more general form

$$u(x,y,t) = U\{x,y,t,w[z(x,y,t)]\}, \qquad (3)$$

where $\alpha(x,y,t)$, $\beta(x,y,t)$, $z(x,y,t)$ and $w(z)$ are all differentiable functions to be determined (proof ignored). We will consider the interesting case of $\beta \neq 0$ only. (Or else, if $\beta = 0$, Ansatz 2 would turn out to be trivial, based on which we could not get any similarity reduction.)

With the help of computerized symbolic computation, we substitute (2) into (1), to get

$$\begin{aligned}
&2 w' \beta_y z_y + \beta w'' {z_y}^2 + \alpha_{yy} + w \beta_{yy} + \beta w' z_{yy} + {\alpha_x}^2 \\
&+ w' z_t \beta_x + 2 w \alpha_x \beta_x + w^2 {\beta_x}^2 + w' \beta_t z_x \\
&+ \beta w'' z_t z_x + 2 \beta w' \alpha_x z_x + 2 \alpha w' \beta_x z_x \\
&+ 4 \beta w w' \beta_x z_x + \beta^2 {w'}^2 {z_x}^2 + \alpha \beta w'' {z_x}^2 \\
&+ \beta^2 w w'' {z_x}^2 + 4 s w''' \beta_x {z_x}^3 + s \beta w'''' {z_x}^4 \\
&+ 6 w''''' \beta_x {z_x}^5 + \beta w'''''' {z_x}^6 + \alpha_{xt} + w \beta_{xt} \\
&+ \beta w' z_{xt} + \alpha \alpha_{xx} + \beta w \alpha_{xx} + \alpha w \beta_{xx} \\
&+ \beta w^2 \beta_{xx} + 6 s w'' {z_x}^2 \beta_{xx} + 15 w'''' {z_x}^4 \beta_{xx} \\
&+ \alpha \beta w' z_{xx} + \beta^2 w w' z_{xx} + 12 s w'' \beta_x z_x z_{xx} \\
&+ 6 s \beta w''' {z_x}^2 z_{xx} + 60 w'''' \beta_x {z_x}^3 z_{xx} \\
&+ 15 \beta w''''' {z_x}^4 z_{xx} + 6 s w' \beta_{xx} z_{xx} \\
&+ 90 w''' {z_x}^2 \beta_{xx} z_{xx} + 3 s \beta w'' {z_{xx}}^2 \\
&+ 90 w''' \beta_x z_x {z_{xx}}^2 + 45 \beta w'''' {z_x}^2 {z_{xx}}^2 \\
&+ 45 w'' \beta_{xx} {z_{xx}}^2 + 15 \beta w''' {z_{xx}}^3 + 4 s w' z_x \beta_{xxx} \\
&+ 20 w''' {z_x}^3 \beta_{xxx} + 60 w'' z_x z_{xx} \beta_{xxx} \\
&+ 4 s w' \beta_x z_{xxx} + 4 s \beta w'' z_x z_{xxx} \\
&+ 60 w''' \beta_x {z_x}^2 z_{xxx} + 20 \beta w'''' {z_x}^3 z_{xxx} \\
&+ 60 w'' z_x \beta_{xx} z_{xxx} + 60 w'' \beta_x z_{xx} z_{xxx} \\
&+ 60 \beta w''' z_x z_{xx} z_{xxx} + 20 w' \beta_{xxx} z_{xxx} \\
&+ 10 \beta w'' {z_{xxx}}^2 + s \alpha_{xxxx} + s w \beta_{xxxx} \\
&+ 15 w'' {z_x}^2 \beta_{xxxx} + 15 w' z_{xx} \beta_{xxxx} \\
&+ s \beta w' z_{xxxx} + 30 w'' \beta_x z_x z_{xxxx} \\
&+ 15 \beta w''' {z_x}^2 z_{xxxx} + 15 w' \beta_{xx} z_{xxxx} \\
&+ 15 \beta w'' z_{xx} z_{xxxx} + 6 w' z_x \beta_{xxxxx} \\
&+ 6 w' \beta_x z_{xxxxx} + 6 \beta w'' z_x z_{xxxxx} + \alpha_{xxxxxx} \\
&+ w \beta_{xxxxxx} + \beta w' z_{xxxxxx} = 0.
\end{aligned} \qquad (4)$$

When we demand that (4) be an ordinary differential equation (ODE) for $w(z)$, the ratios of the coefficients of different derivatives and powers of $w(z)$ have to be some functions of z only. To do so, there exists a couple of cases to be discussed separately, as follows:

Case A – The First Reduction: $z_x \neq 0$

For (4), we now use the coefficient of w'''''', i.e., $\beta {z_x}^6$, as the normalizing coefficient, and after the simplifications get

$$6 \beta_x z_x + \beta \Gamma_1(z) {z_x}^2 + 15 \beta z_{xx} = 0\ , \qquad (5)$$

$$\begin{aligned}
&s \beta {z_x}^2 + \beta \Gamma_2(z) {z_x}^4 + 15 {z_x}^2 \beta_{xx} + 60 \beta_x z_x z_{xx} \\
&\quad + 45 \beta {z_{xx}}^2 + 20 \beta z_x z_{xxx} = 0\ ,
\end{aligned} \qquad (6)$$

$$\begin{aligned}
&4 s \beta_x {z_x}^3 + \beta \Gamma_3(z) {z_x}^6 + 6 s \beta {z_x}^2 z_{xx} \\
&\quad + 90 {z_x}^2 \beta_{xx} z_{xx} + 90 \beta_x z_x {z_{xx}}^2 + 15 \beta {z_{xx}}^3 \\
&\quad + 20 {z_x}^3 \beta_{xxx} + 60 \beta_x {z_x}^2 z_{xxx} \\
&\quad + 60 \beta z_x z_{xx} z_{xxx} + 15 \beta {z_x}^2 z_{xxxx} = 0\ ,
\end{aligned} \qquad (7)$$

$$\beta + \Gamma_4(z) {z_x}^4 = 0\ , \qquad (8)$$

$$\begin{aligned}
&\beta {z_y}^2 + \beta z_t z_x + \alpha \beta {z_x}^2 + \beta \Gamma_5(z) {z_x}^6 \\
&\quad + 6 s {z_x}^2 \beta_{xx} + 12 s \beta_x z_x z_{xx} + 3 s \beta {z_{xx}}^2
\end{aligned}$$

$$\begin{aligned}&+45\beta_{xx}z_{xx}^2+60z_x z_{xx}\beta_{xxx}+4s\beta z_x z_{xxx}\\&+60z_x\beta_{xx}z_{xxx}+60\beta_x z_{xx}z_{xxx}+10\beta z_{xxx}^2\\&+15z_x^2\beta_{xxxx}+30\beta_x z_x z_{xxxx}+15\beta z_{xx}z_{xxxx}\\&+6\beta z_x z_{xxxxx}=0\ ,\end{aligned}\tag{9}$$

$$4\beta_x z_x+\Gamma_6(z)z_x^6+\beta z_{xx}=0\ ,\tag{10}$$

$$\begin{aligned}&2\beta_y z_y+\beta z_{yy}+z_t\beta_x+\beta_t z_x+2\beta\alpha_x z_x\\&+2\alpha\beta_x z_x+\beta\Gamma_7(z)z_x^6+\beta z_{xt}+\alpha\beta z_{xx}\\&+6s\beta_{xx}z_{xx}+4sz_x\beta_{xxx}+4s\beta_x z_{xxx}\\&+20\beta_{xxx}z_{xxx}+15z_{xx}\beta_{xxxx}+s\beta z_{xxxx}\\&+15\beta_{xx}z_{xxxx}+6z_x\beta_{xxxxx}+6\beta_x z_{xxxxx}\\&+\beta z_{xxxxxx}=0\ ,\end{aligned}\tag{11}$$

$$\beta+\Gamma_8(z)z_x^4=0\ ,\tag{12}$$

$$\beta_x^2+\beta\Gamma_9(z)z_x^6+\beta\beta_{xx}=0\ ,\tag{13}$$

$$\begin{aligned}&\beta_{yy}+2\alpha_x\beta_x+\beta\Gamma_{10}(z)z_x^6+\beta_{xt}+\beta\alpha_{xx}\\&+\alpha\beta_{xx}+s\beta_{xxxx}+\beta_{xxxxxx}=0\ ,\end{aligned}\tag{14}$$

$$\begin{aligned}&\alpha_{yy}+\alpha_x^2+\beta\Gamma_{11}(z)z_x^6+\alpha_{xt}+\alpha\alpha_{xx}\\&+s\alpha_{xxxx}+\alpha_{xxxxxx}=0\ ,\end{aligned}\tag{15}$$

with $\Gamma_i(z)$'s as some functions of z to be determined, which represent the ratios of the coefficients of different derivatives and powers of $w(z)$ in (4), assumed functions of z only, to make (4) as an ODE for $w(z)$.

Searching for the formats of $\alpha(x,y,t)$, $\beta(x,y,t)$, $z(x,y,t)$ and $\Gamma_i(z)$'s, we are able to use the following remarks without loss of generality [21]:

Remark 1: We reserve uppercased greek letters for the functions of z to be determined, so that after such operations as differentiation, integration, exponentiation, rescaling, etc., the result can be denoted by the same letter.

Remark 2: In determinating $\alpha(x,y,t)$, $\beta(x,y,t)$, $z(x,y,t)$ and $w(z)$, we are able to exploit three freedoms, without loss of generality: 1) if $\alpha(x,y,t)$ has the form $\alpha(x,y,t)=\alpha_0(x,y,t)+\beta(x,y,t)\Omega(z)$, we can take $\Omega\equiv 0$ [by substituting $w(z)\to w(z)-\Omega(z)$]; 2) if $\beta(x,y,t)$ has the form $\beta(x,y,t)=\beta_0(x,y,t)\Omega(z)$, then we can take $\Omega\equiv 1$ [by substituting $w(z)\to w(z)/\Omega(z)$]; and 3) if $z(x,y,t)$ is determined by an equation of the form $\Omega(z)=z_0(x,y,t)$, where $\Omega(z)$ is any invertible function, then we can take $\Omega(z)=z$ [by substituting $z\to\Omega^{-1}(z)$]. Each freedom of the three variables $\alpha(x,y,t)$, $\beta(x,y,t)$ and $z(x,y,t)$ can be used just once in the way of calculations to avoid the loss of generality.

Combining (8) and (12) with the second freedom in Remark 2, we get immediately

$$\beta=z_x^4\ ,\qquad \Gamma_4=\Gamma_8=-1\ .\tag{16}$$

Substituting (16) into (5) and using the Remarks 1 and 2 yields

$$z=\theta(y,t)x+\phi(y,t)\text{ with }\theta(y,t)\neq 0;\quad \Gamma_1=0.\tag{17}$$

Then, substituting (16) and (17) into (6) and using the Remarks 1 and 2 leads to

$$\theta=\text{constant}\neq 0\ ;\qquad \Gamma_2=-\frac{s}{\theta^2}\ .\tag{18}$$

Equations (7), (10) and (13) reduce to

$$\Gamma_3=\Gamma_6=\Gamma_9=0\ .\tag{19}$$

So far only four equations, i.e., (9), (11), (14) and (15) are left, which after simplifications look like:

$$\theta^2\alpha+\theta^6\Gamma_5+\theta\phi_t+\phi_y^2=0,\tag{20}$$

$$\theta^6\Gamma_7+\phi_{yy}+2\theta\alpha_x=0,\tag{21}$$

$$\theta^6\Gamma_{10}+\alpha_{xx}=0,\tag{22}$$

$$\begin{aligned}&\theta^{10}\Gamma_{11}+\alpha_{yy}+\alpha_x^2+\alpha_{xt}+\alpha\alpha_{xx}\\&+s\alpha_{xxxx}+\alpha_{xxxxxx}=0.\end{aligned}\tag{23}$$

With the Remarks 1 and 2, (22) gives rise to

$$a(x,y,t)=\lambda(y,t)+x\sigma(y,t)\ ,\qquad \Gamma_{10}=0\ ,\tag{24}$$

where $\lambda(y,t)$ and $\sigma(y,t)$ are a couple of differentiable functions.

Substituting (24) into (20), (21) and (23) yields

$$\theta^2\lambda+\theta^2x\sigma+\theta\phi_t+\phi_y^2=-\theta^6\Gamma_5(z)\ ,\tag{25}$$

$$2\theta\sigma+\phi_{yy}=-\theta^6\Gamma_7(z)\ ,\tag{26}$$

$$\sigma^2+\sigma_t+\lambda_{yy}+x\sigma_{yy}=-\theta^{10}\Gamma_{11}(z)\ .\tag{27}$$

Since $z=\theta x+\phi(y,t)$ with $\theta\neq 0$, but the left-hand side of (26) is only a function of y and t, we should have

$$\Gamma_7(z)=\rho\ ,\tag{28}$$

where ρ is a constant. Similar investigations on (25) and (27) yield

$$\Gamma_5(z) = \zeta z + \eta \ , \qquad \Gamma_{11}(z) = v z + \tau \ , \tag{29}$$

where ζ, η, v and τ are also constants. In (26) and (27), along with (28) and (29), we equate to zero the coefficients of like powers of x, and with the integration with respect to t obtain

$$\sigma(y,t) = -\theta^5 \zeta = \text{const} \ , \tag{30}$$

$$v = 0 \ , \tag{31}$$

$$\phi(y,t) = -\frac{\theta^6 y^2}{2} (\rho - 2\zeta) + \chi(t) + y\,\upsilon(t) \ , \tag{32}$$

$$\lambda(y,t) = -\frac{\theta^{10} y^2}{2} \left(\tau + \zeta^2\right) + \delta(t) + y\,\mu(t) \ , \tag{33}$$

where $\chi(t)$, $\upsilon(t)$, $\delta(t)$ and $\mu(t)$ are all differentiable functions. Similarly, in (25) with Expression (29), equating to zero the coefficients of like powers of y leads to

$$\tau = 2\rho^2 - 9\rho\zeta + 9\zeta^2 \ , \tag{34}$$

$$\mu(t) = 2\rho\,\theta^4\,\upsilon(t) - 5\,\theta^4\,\zeta\,\upsilon(t) - \frac{\upsilon'(t)}{\theta} \ , \tag{35}$$

$$\delta(t) = -\eta\,\theta^4 - \theta^4\,\zeta\,\chi(t) - \frac{\upsilon(t)^2}{\theta^2} - \frac{\chi'(t)}{\theta} \ . \tag{36}$$

Therefore, we have obtained *the first set of similarity reductions* of (1), which is

$$\begin{aligned} u^{(I)}(x,y,t) = \\ & -\eta\,\theta^4 - \frac{\theta^{10} y^2}{2} (2\rho - 5\zeta)(\rho - 2\zeta) \\ & -\theta^5 x \zeta - \theta^4 \zeta\,\chi(t) - \frac{\upsilon(t)^2}{\theta^2} - \frac{\chi'(t)}{\theta} \\ & + y\left[2\rho\,\theta^4\,\upsilon(t) - 5\,\theta^4\,\zeta\,\upsilon(t) - \frac{\upsilon'(t)}{\theta}\right] \\ & + \theta^4 w\left[z^{(I)}(x,y,t)\right] , \end{aligned} \tag{37}$$

where

$$z^{(I)}(x,y,t) = \theta x - \frac{\theta^6 y^2}{2} (\rho - 2\zeta) + \chi(t) + y\,\upsilon(t) \tag{38}$$

and $w(z)$ must satisfy the following 6th order ODE

$$\begin{aligned} & 2\rho^2\theta^2 - 9\rho\,\theta^2\zeta + 9\,\theta^2\zeta^2 + \rho\,\theta^2 w' - \theta^2 w'^2 \\ & + \theta^2(\eta + \zeta z)w'' - \theta^2 w w'' - s w'''' - \theta^2 w'''''' = 0. \end{aligned} \tag{39}$$

Solution (37), without the last term, appears as the *polynomial in x and y, with certain arbitrary functions of t involved*. On the other hand, the expression of the *last term*, along with (38), remains an open question for the time being until the properties of the above 6th order ODE have been investigated.

Case B – Two More Reductions: $z_x = 0$

We use the coefficient of w'', i.e. $\beta z_y{}^2$, as the normalizing coefficient for (4), with the assumption that $z_y \neq 0$, and after simplifications get

$$\beta\,\Gamma_1(z) z_y{}^2 + \beta z_{yy} + 2 z_y \beta_y + z_t \beta_x = 0 \ , \tag{40}$$

$$\beta\,\Gamma_2(z) z_y{}^2 + \beta_x{}^2 + \beta\,\beta_{xx} = 0 \ , \tag{41}$$

$$\begin{aligned} & \beta\,\Gamma_3(z) z_y{}^2 + \beta_{yy} + 2\,\alpha_x \beta_x + \beta_{xt} \\ & + \beta\,\alpha_{xx} + \alpha\,\beta_{xx} + s\,\beta_{xxxx} + \beta_{xxxxxx} = 0 \ , \end{aligned} \tag{42}$$

$$\begin{aligned} & \beta\,\Gamma_4(z) z_y{}^2 + \alpha_{yy} + \alpha_x{}^2 + \alpha_{xt} + \alpha\,\alpha_{xx} \\ & + s\,\alpha_{xxxx} + \alpha_{xxxxxx} = 0 \ , \end{aligned} \tag{43}$$

with $\Gamma_i(z)$'s as some functions of z to be determined.

An observation on (41) indicates that a convenient assumption is

$$\beta = \beta(y,t) \qquad \text{with} \qquad \Gamma_2 = 0 \ . \tag{44}$$

Equation (40) is divided by $\beta z_y{}^2$, integrated with respect to y, and combined with Remark 1 and the second freedom in Remark 2, yielding

$$z = \rho(t) \int \beta(y,t)^{-2}\,\mathrm{d}y + \sigma(t) \text{ with } \Gamma_1 = 0, \tag{45}$$

where $\rho(t) \neq 0$ and $\sigma(t)$ are a couple of differentiable functions.

Similarly, (42) is divided by β^4, integrated twice with respect to x, and combined with the first freedom in Remark 2, giving rise to

$$\alpha(x,y,t) = x\,\eta(y,t) + \mu(y,t) - \frac{x^2 \beta_{yy}(y,t)}{2\beta(y,t)} \quad \text{with } \Gamma_3 = 0, \tag{46}$$

where $\eta(y,t)$ and $\mu(y,t)$ are a couple of differentiable functions.

So far, (43) becomes

$$\begin{aligned} & 2\beta^3\eta^2 + 2\Gamma_4(z)\rho^2 + 2\beta^3\eta_t - 6x\beta^2\eta\,\beta_{yy} \\ & - 2\beta^2\mu\,\beta_{yy} + 2x\beta\,\beta_t\,\beta_{yy} - 2x^2\beta_y{}^2\beta_{yy} \\ & + 4x^2\beta\,\beta_{yy}{}^2 + 2x\beta^3\eta_{yy} + 2\beta^3\mu_{yy} - 2x\beta^2\beta_{yyt} \\ & + 2x^2\beta\,\beta_y\,\beta_{yyy} - x^2\beta^2\beta_{yyyy} = 0, \end{aligned} \tag{47}$$

which, when we equate to zero the coefficients of like powers of x, can be separated out as

$$x^2 : 4\beta{\beta_{yy}}^2 - 2{\beta_y}^2\beta_{yy} + 2\beta\beta_y\beta_{yyy} - \beta^2\beta_{yyyy} = 0, \quad (48)$$

$$x^1 : \beta_t\,\beta_{yy} - 3\beta\,\eta\,\beta_{yy} + \beta^2\,\eta_{yy} - \beta\,\beta_{yyt} = 0, \quad (49)$$

$$x^0 : \beta^3\eta^2 + \Gamma_4(z)\rho^2 + \beta^3\eta_t - \beta^2\mu\beta_{yy} + \beta^3\mu_{yy} = 0. \quad (50)$$

There again exists a couple of subcases to be discussed respectively, as follows:

Reduction II: $z_x = 0$ and $\beta_y \neq 0$

Equation (48) is integrated with respect to y (with a vanishing constant of integration), divided by β, integrated again with respect to y (with a vanishing constant of integration), divided by β_y, integrated the third time with respect to y, and transformed to its exponential form with the fourth-time integration with respect to y, so that

$$\beta(y,t) = \frac{1}{\delta(t) + y\lambda(t)} \ , \quad (51)$$

where $\delta(t)$ and $\lambda(t)$ are a couple of non-zero differentiable functions. Then, in (49) we can assume that

$$\eta(y,t) = \varepsilon(t)\,\beta(y,t) \ , \quad (52)$$

where $\varepsilon(t)$ can be calculated as

$$\varepsilon(t) = \delta'(t) - \frac{\delta(t)\,\lambda'(t)}{\lambda(t)} \ . \quad (53)$$

Equation (50) now reduces to

$$\Gamma_4(z) - \frac{2\lambda(t)^2\mu(y,t)}{[\delta(t)+y\lambda(t)]^5\rho(t)^2} + \frac{\mu_{yy}(y,t)}{[\delta(t)+y\lambda(t)]^3\rho(t)^2} - \Big\{2\lambda(t)\delta'(t)\lambda'(t) - 2\delta(t)\lambda'(t)^2 - \lambda(t)^2\delta''(t) + \delta(t)\lambda(t)\lambda''(t)\Big\}\Big\{\lambda(t)^2\,[\delta(t)+y\lambda(t)]^4\rho(t)^2\Big\}^{-1} = 0, \quad (54)$$

the observations on which indicate that we are able to assume that

$$\mu(y,t) = \frac{\psi(t)}{\beta(y,t)}, \quad (55)$$

$$\Gamma_4(z) = \nu z^{-\frac{4}{3}}, \quad (56)$$

$$\sigma(t) = 0, \quad (57)$$

with ν defined as a constant and $\psi(t)$ calculated as

$$\psi(t) = \frac{3^{\frac{4}{3}}\,\nu\,\lambda(t)^{\frac{10}{3}}\,\rho(t)^{\frac{2}{3}} + 2\,\delta(t)\,\lambda'(t)^2 + \lambda(t)^2\,\delta''(t) - \lambda(t)\,[2\,\delta'(t)\,\lambda'(t) + \delta(t)\,\lambda''(t)]}{2\,\lambda(t)^4}. \quad (58)$$

In fact, to this stage we have seen *the second set of the similarity reductions* of (1), with the resulting ODE for $w(z)$ as

$$w''(z) - \nu z^{-\frac{4}{3}} = 0 \ , \quad (59)$$

which has been reduced from (4). Furthermore, this ODE can be directly integrated out as

$$w(z) = \omega - \frac{9}{2}\,\nu z^{\frac{2}{3}} + \phi z, \quad (60)$$

where ω and ϕ are a couple of constants of integration.

Putting everything together, we hereby present *a family of the exact analytic solutions* for (1), written as

$$u^{(II)}(x,y,t) = -\frac{x^2\lambda(t)^2}{[\delta(t)+y\lambda(t)]^2} + \frac{x\left[\delta'(t) - \frac{\delta(t)\lambda'(t)}{\lambda(t)}\right]}{\delta(t)+y\lambda(t)} + \frac{\omega}{\delta(t)+y\lambda(t)} + \frac{\phi\,[\delta(t)+y\lambda(t)]^2\rho(t)}{3\lambda(t)} + \frac{\delta(t)+y\lambda(t)}{2\lambda(t)^4}\Big\{2\,\delta(t)\,\lambda'(t)^2 + \lambda(t)^2\,\delta''(t) - \lambda(t)\,\big[2\,\delta'(t)\,\lambda'(t) + \delta(t)\,\lambda''(t)\big]\Big\}, \quad (61)$$

which has its *rational expression with respect to x and y, with certain arbitrary functions of t involved*.

Reduction III: $z_x = 0$ and $\beta_y = 0$

This time, (48) is satisfied automatically, while (49) can be integrated out as

$$\eta(y,t) = \delta(t) + y\lambda(t) \ , \tag{62}$$

where $\delta(t)$ and $\lambda(t)$ are a couple of differentiable functions.

Then (50) is integrated twice with suitable transformations, Remark 1 and the second freedom in Remark 2, yielding

$$\mu(y,t) = -\frac{1}{12\beta(t)}\Big\{12\beta(t)^2\phi(t) + 12y\psi(t)\rho(t) + y^2\beta(t)\Big[6\delta(t)^2 + 4y\delta(t)\lambda(t) + y^2\lambda(t)^2 + 6\,\delta'(t) + 2y\,\lambda'(t)\Big]\Big\}, \tag{63}$$

$$\Gamma_4(z) = 0 \ . \tag{64}$$

Reduced from (4), the resulting ODE for $w(z)$ comes out as

$$w''(z) = 0 \ , \tag{65}$$

which can be solved out, i. e.,

$$w(z) = \nu z + \kappa \ , \tag{66}$$

where ν and κ are a couple of constants of integration.

Hence, we have obtained *the third set of the similarity reductions*, or *another family of the exact analytic solutions* for (1), as

$$u^{(III)}(x,y,t) = x\,[\delta(t) + y\,\lambda(t)] + \beta(t)\,[\kappa - \phi(t) + \nu\,\sigma(t)] + \frac{y\,[\nu - \psi(t)]\,\rho(t)}{\beta(t)} - \frac{y^2\,[\delta(t)^2 + \delta'(t)]}{2} - \frac{y^3\,[2\,\delta(t)\,\lambda(t) + \lambda'(t)]}{6} - \frac{y^4\,\lambda(t)^2}{12}, \tag{67}$$

which is also *rational with respect to x and y, with certain arbitrary functions of t involved*.

Three different similarity reductions have been found, corresponding to the aforementioned three different cases, which are all the possibilities that could be explored via the Clarkson-Kruskal method.

Conclusions

Symbolic computation is a new branch of artificial intelligence, with its remarkable feature as the permeation of computer sciences among various fields of science and engineering. As to such liquid surfaces as those for oceans, rivers, liquid propellant for rockets and aviation kerosene, people have devoted much effort. In this paper we have studied the Hărăgus-Courcelle-Il'ichev model for the surface of liquid waves, in the presence of surface tension or sea ice, with computerized symbolic computation. The results are the above sets of the similarity reductions, some of which are explicitly written out as the exact analytic solutions having their rational expressions with respect to the dimensionless spatial variables x and y.

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